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ON THE STRONG LAW OF LARGE NUMBERS AND RELATED RESULTS FOR QUAS--ETC(U)

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ON THE STRONG LAW OF LARGE NUMBERS AND RELATED RESULTS FOR QUASI-STATIONARY SEQUENCES

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ABSTRACT

ON THE STRONG LAW OF LARGE NUMBERS AND RELATED RESULTS FOR QUASI-STATIONARY SEQUENCES

Under second moment assumptions and weak dependence conditions on a sequence of random variables $\{X_i\}$, Gapoškin (1975) has established almost sure convergence of the series $\sum_1^\infty \lambda_k X_k$ under certain restrictions on the rate of convergence to 0 of the constants $\{c_k\}$. Similarly, Móricz (1977) has established conditions for the almost sure convergence to 0 of the sequence $\lambda_n \sum_1^n X_k$. In the present paper, some extensions of these results are obtained.

1. <u>Main results and discussion</u>. Consider a sequence of random variables $\{X_i\}$ satisfying

$$EX_i \equiv 0 \ , \quad EX_i^2 \equiv 1$$

and, for a sequence of constants $\{\phi_i\}$,

$$|EX_j X_k| \le \phi_{k-j}, \text{ all } j \le k.$$

Such a sequence $\{X_i\}$ is called *quasi-stationary* with respect to the sequence $\{\phi_i\}$. The almost sure asymptotic behavior of the sum $\sum_{i=1}^n X_i$ may be characterized by an assertion of the form

$$\text{(A)} \qquad \lambda_n \sum_{i=1}^{n} X_i \to 0 \ , \quad n \to \infty \ , \quad \text{w.p.1},$$

where $\{\lambda_n\}$ is a sequence of positive constants tending to 0. It is of interest to establish (A) under mild restrictions on the constants $\{\phi_i\}$ and $\{\lambda_i\}$. A related problem concerns the almost sure behavior of the sum $\sum_{i=1}^n \lambda_i X_i$ for such a sequence of constants $\{\lambda_n\}$. In this case the desired assertion is

$$(\alpha) \qquad \sum_{i=1}^{\infty} \lambda_i X_i \ \text{ converges w.p.1.}$$

By the well-known Kronecker lemma, (α) implies (A) in the case of λ_n non-increasing.

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Rademacher (1922) and Mensov (1923) independently established that (α) holds if

(1.1a) $\phi_i = 0$, $i > 0$,

and

(1.1b) $\sum_1^\infty \lambda_n^2 \log^2 n < \infty$.

Kac, Salem and Zygmund (1948) relaxed (1.1a) to $\phi_n = O(n^{-1-\epsilon})$ for an $\epsilon > 0$. Gapoškin (1975) proved the following much broader result. Put

$$w(n) = \sum_{i=1}^n \phi_i \ .$$

THEOREM α (Gapoškin). If

(1.2) $\sum_1^\infty w(n) \lambda_n^2 \log^2 n < \infty$,

then (α) *holds*.

This theorem allows the possibility of $w(n) \to \infty$, whereas the earlier results are confined to the case $w(\infty) < \infty$.

Returning to (A), we have

COROLLARY α. *If* (1.2) *is satisfied and* λ_n *is nonincreasing, then* (A) *holds*.

On the other hand, a direct approach — bypassing (α) — offers the possibility of obtaining (A) under weaker restrictions than (1.2). In this direction, Móricz (1977) has obtained the following result.

THEOREM A (Móricz). *If*

(1.3a) $\quad \sum_1^\infty w(n)\,\lambda_n^2 < \infty$,

(1.3b) $\quad w(n)\,\lambda_n^2$ *is nonincreasing,*

and

(1.3c) $\quad w(2n)/w(n) \geq q > 1$, *all* n,

then (A) *holds.*

Note that (1.3a) relaxes (1.2). However, (1.3c) requires $w(n)$ to grow at a fast rate. For example, (1.3c) is satisfied by $w(n)$ of the form $w(n) = cn^a$, but not by $w(n)$ of the form $w(n) = \exp(2\sqrt{\log n})$. For the latter, Theorem A is inapplicable, whereas Theorem α does yield a conclusion.

The present note provides an alternate to Theorem A which essentially removes condition (1.3c). As in [5], put $W(1) = w(1)$ and, for $n \geq 2$, define $W(n)$ by

$$W^{\frac{1}{2}}(n) = W^{\frac{1}{2}}([\tfrac{1}{2}n] - 1) + w^{\frac{1}{2}}([\tfrac{1}{2}n]) .$$

THEOREM B. *If*

(1.4a) $\quad \sum_1^\infty W(n)\,\lambda_n^2 < \infty$

and

(1.4b) $W(n)\,\lambda_n^2$ *is nonincreasing,*

then (A) *holds.*

Conditions (1.3a) and (1.3c) together imply (1.4a), as evident from the Lemma below. Also, the mild constraints (1.3b) and (1.4b) are mere variants of each other. Thus Theorem B has somewhat broader application than Theorem A. In particular, it yields

EXAMPLE. Consider $w(n) = \exp(2\sqrt{\log n})$. In this case (by the Lemma below)

$$W(n) = O(w(n) \log n) \ ,$$

so that (A) holds if λ_n satisfies (1.4b) *for this* $w(n)$ and if

(1.5) $\sum_1^\infty w(n)\,\lambda_n^2 \log n < \infty$. □

In the preceding example, the use of Corollary α would be less effective than Theorem B, since (1.5) is weaker than (1.2). The gain in effectiveness of Theorem B over Corollary α occurs when $w(n)$ grows sufficiently fast.

LEMMA. (i) *In general,* $W(n) = O(w(n) \log^2 n)$.

(ii) *If* $w(n) = \exp(2\sqrt{\log n})$, *then* $W(n) = O(w(n) \log n)$.

(iii) *If* $w(n)$ *satisfies* (1.5c), *then* $W(n) = O(w(n))$.

As a complement to Theorem B, the following generalization of Theorem α will be established.

THEOREM β. *If* (1.4a) *and*

(1.6a) $\quad \sum_1^\infty w(n)\, \lambda_n^2 (\log n)(\log\log n)^{1+\epsilon} < \infty$, *for some* $\epsilon > 0$,

are satisfied, then (α) *holds.*

Since (1.2) implies each of (1.4a) and (1.6a), Theorem β generalizes Theorem α.

2. Proofs.

PROOF OF THE LEMMA. Note that, for $2^k \le n < 2^{k+1}$,

(2.1) $\quad W^{\frac{1}{2}}(n) < W^{\frac{1}{2}}(2^{k+1} - 1) = \sum_{j=0}^k w^{\frac{1}{2}}(2^j)$.

Thus $W^{\frac{1}{2}}(n) \le kw^{\frac{1}{2}}(n) = O(w^{\frac{1}{2}}(n)\log n)$, which gives (i). Now, for $j \le k$,

$$\begin{aligned}
\exp\sqrt{j} &= \exp\sqrt{k}\, \exp\left(\frac{j - k}{\sqrt{j} + \sqrt{k}}\right) \\
&\le \exp\sqrt{k}\, \exp\left(\frac{j - k}{2\sqrt{k}}\right) \\
&= \exp \tfrac{1}{2}\sqrt{k} \left(\exp \frac{1}{2\sqrt{k}}\right)^j .
\end{aligned}$$

Thus by (2.1) we obtain, for the case $w(n) = \exp(2\sqrt{\log n})$, that

$$W^{\frac{1}{2}}(n) \le (\exp \tfrac{1}{2}\sqrt{k})\, \frac{\exp\sqrt{k}}{\exp\left(\frac{1}{2\sqrt{k}}\right) - 1} \le 2\sqrt{k}\, \exp\sqrt{k} ,$$

i.e., $W^{\frac{1}{2}}(n) = O(\sqrt{\log n}\, w^{\frac{1}{2}}(n))$, so that (ii) is proved. Finally, for $w(n)$ satisfying (1.3c), the use of (2.1) yields

$$W^{\frac{1}{2}}(n) \le W^{\frac{1}{2}}(2^k) \sum_{j=0}^{k} \left(\frac{1}{q}\right)^{\frac{1}{2}(k-j)} ,$$

from which (iii) follows. □

In proving Theorems B and β, the following maximal inequality will be used.

LEMMA 2.1. *For* $m \ge 1$, $n \ge 1$,

$$E\left\{\max_{1\le k\le n} \left(\sum_{i=m+1}^{m+k} a_i X_i\right)^2\right\} \le 2W(n) \sum_{i=m+1}^{m+k} a_i^2 .$$

This was proved by Móricz (1976), extending an earlier result of Serfling (1970a). For Theorem β, we will also need the following easily proved parallel result [5], [8].

LEMMA 2.2. *For* $n \ge 1$,

$$E\left\{\left(\sum_{i=m+1}^{m+n} a_i X_i\right)^2\right\} \le 2w(n) \sum_{i=m+1}^{m+n} a_i^2 .$$

PROOF OF THEOREM B. In order to show (A), it is equivalent to show that for every $\epsilon > 0$,

(2.2) $\quad P\{|\lambda_n S_n| > \epsilon \text{ infinitely often}\} = 0.$

Now observe that the nonincreasingness of $\lambda_n^2 W(n)$, combined with the non-decreasingness of $W(n)$, implies that λ_n is nondecreasing. Thus, by the Borel-Cantelli lemma, (2.2) holds if

$$(2.3) \qquad \sum_{k=0}^{\infty} P\{\lambda_{2^k} \max_{2^k \le n < 2^{k+1}} |S_n| > \epsilon\} < \infty .$$

Now

$$P\{\lambda_{2^k} \max_{2^k \le n < 2^{k+1}} |S_n| > \epsilon\}$$

$$\le \epsilon^{-2} \lambda_{2^k}^2 E\{\max_{2^k \le n < 2^{k+1}} S_n^2\}$$

$$\le 2\epsilon^{-2} \lambda_{2^k}^2 E\{S_{2^k}^2 + \max_{2^k \le n < 2^{k+1}} (S_n - S_{2^k})^2\} .$$

A two-fold application of Lemma 2.1 gives

$$P\{\lambda_{2^k} \max_{2^k \le n < 2^{k+1}} |S_n| > \epsilon\} \le 4\lambda_{2^k}^2 W(2^k) 2^k .$$

Thus the sum in (2.3) is bounded by $4 \sum_{k=0}^{\infty} 2^k \lambda_{2^k}^2 W(2^k)$, which in turn in view of (1.3b) is clearly bounded by $4 \sum_1^{\infty} \lambda_n^2 W(n)$. By (1.3a), the required (2.3) thus holds. □

PROOF OF THEOREM β. Following the approach of [1], and using a standard elementary argument, we first establish that T_{2^k} converges w.p.1 to a limit T_∞ , by showing that

$$(2.4) \qquad \sum_{k=0}^{\infty} ||\Delta_k|| < \infty ,$$

where $\Delta_k = S_{2^{k+1}} - S_{2^k}$ and $||\Delta_k||$ denotes $(E\,\Delta_k^2)^{1/2}$. By the Cauchy-Schwarz inequality,

$$\sum ||\Delta_k|| \leq \left\{\sum d_k^2 ||\Delta_k||^2\right\}\left\{\sum d_k^{-2}\right\}$$

for positive constants d_k. Choose $d_k = k^{\frac{1}{2}} (\log k)^{\frac{1}{2}(1+\epsilon)}$, $\epsilon > 0$. Then, applying Lemma 2.2 and (1.6a), we obtain for an appropriate constant C,

$$\sum ||\Delta_k|| \leq C \sum k(\log k)^{1+\epsilon} \, [2w(2^k) \sum_{j=2^k+1}^{2^{k+1}} \lambda_j^2]$$

$$\leq 2C \sum_{n=1}^{\infty} (\log n)(\log \log n)^{1+\epsilon} \, w(n) \, \lambda_n^2 < \infty .$$

Next we establish that T_n converges w.p.1 to T_∞, by showing that

$$\max_{2^k \leq n < 2^{k+1}} |T_n - T_{2^k}| \longrightarrow 0 \ \text{ w.p.1.}$$

This follows, by an argument similar to the proof of Theorem B, if

$$\sum_{k=0}^{\infty} E\left\{\max_{2^k \leq n < 2^{k+1}} (T_n - T_{2^k})^2\right\} < \infty ,$$

which in turn is established by Lemma 2.2 and (1.4a), via

$$2 \sum_{k=0}^{\infty} W(2^k) \sum_{n=2^k}^{2^{k+1}-1} \lambda_n^2 \leq \sum_{n=1}^{\infty} W(n) \, \lambda_n^2 < \infty . \quad \square$$

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Under second moment assumptions and weak dependence conditions on a sequence of random variables $\{X_i\}$, Gapoškin (1975) has established almost sure convergence of the series $\sum_1^\infty \lambda_k X_k$ under certain restrictions on the rate of convergence to 0 of the constants $\{c_k\}$. Similarly, Móricz (1977) has established conditions for the almost sure convergence to 0 of the sequence $\lambda_n \sum_1^n X_k$. In the present paper, some extensions of these results are obtained.

<(lambda sub n)(Sum from 1 to n of X sub k)

<c sub k>

Sum from 1 to infinity of lambda sub k X sub k

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